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# TWO-DIMENSIONAL WAVES GENERATED BY A SURFACE PRESSURE DISTURBANCE OVER A SLOPING BEACH

By M. C. Shen

Abstract: Two-dimensional water waves generated by a prescribed pressure disturbance on the equilibrium surface of a uniformly sloping beach is studied by means of the Laplace transform technique applied to the equations of the linear theory. Both transient and steady problems are considered, and in each case the problem reduces to solving an inhomogeneous q-difference equation. The arbitrary constants in the solutions of the transient and the steady-state problem are determined by respectively imposing a boundedness condition and a radiation condition at infinity. The example of an oscillatory point pressure acting on the equilibrium surface of a $45^{\circ}$ beach is worked out in detail. It seems easier to derive the steady-state solution from the transient problem than to find it directly.

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## 1. Introduction

The steady-state problem for waves travelling over a sloping beach free from a pressure disturbance has long been solved by Isaacson [1], Peters [2], [3], and Roseau [4], [5] for general beach angles, using the linear theory of water waves. Recently the beach problem for the case of progressive waves was also solved anew by Lauwerier [6]. An interesting account of the history and late developments of the sloping beach problem can be found in the well-known book "Water Waves" by Stoker [7]. However, the steady-state beach problem of two-dimensional waves let alone the three-dimensional ones, due to a pressure disturbance on the equilibrium surface seems still not well explored, (See [8]) to say nothing of the transient problem. In fact, the transient problem is not as difficult to handle as it would seem to be. One basic method suffices for the solution of both problems. In each case, the method by Peters [3] will be followed to reduce the equations to a single inhomogeneous c-difference equation. Use is made of the solution for the homogeneous equation already constructed in [2], and the solution for the inhomogeneous equation can be found by means of the Plemelj formulas [9], as first used by Peters in the progressive wave problem. The solution for the transient problem, in principle, may be derived from a steady-state problem by imposing a

boundedness condition at infinity.[1] The main difficulty, of course, lies in the inversion of the Laplace transform with respect to time t. The way to solve the steady-state problem is quite straightforward theoretically; but the determination of two arbitrary constants appearing in the solution, by use of the so-called radiation condition, requires an involved calculation.

In Sec. 2 we shall formulate the transient problem in a general setting and investigate in Sec. 3 the a-difference equation derived in Sec. 2. In Sec. 4 we shall consider the example of an oscillatory point pressure acting on the equilibrium surface of a $45^\circ$ beach. The steady-state problem of the above example will be examined in Sec. 5 for general beach angles, and the special case of a $45^\circ$ beach will be discussed in detail in 6. We shall see later that it is really a little easier to derive the steady-state solution from the transient problem as Professor Stoker asserted [7].

## 2. Formulation

Based upon the linear theory, the governing equations for the potential function $\Phi(x,y,t)$ of two-dimensional water waves over

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(1) It is interesting to see that the same situation for the transient problem also appears in other fields. c.f. p.97, [10].

a uniformly sloping beach are, for $t > 0$, $0 < r < \infty$, $0 < \gamma \leq \pi$

$$
(2.1) \qquad
\begin{aligned}
\Phi_{xx} + \Phi_{yy} &= 0, && 0 < \theta < \gamma, \\
\Phi_{tt} + g\Phi_y &= -\frac{1}{\rho} p_t, && \theta = 0, \\
\Phi_n &= 0, && \theta = -\gamma
\end{aligned}
$$

subject to initial conditions,

$$
(2.2) \qquad
\begin{aligned}
\Phi(x,0,0) &= f_1(x), \\
\Phi_t(x,0,0) &= f_2(x).
\end{aligned}
$$

where $(x,y)$ and $(r,\theta)$ are respectively the Cartesian and polar coordinates (Fig. 1) $t$ is the time, $g$ the gravitational constant, $\rho$ the density, the pressure, $n$ the normal direction at $\theta = -\gamma$ and $\gamma$ the beach angle. We assume that $p$ is a prescribed function of $x$ and $t$ and that $\Phi$ and $p_t$ possess Laplace transforms with respect to $x$ and $t$.

We first apply the Laplace transform with respect to $t$

and let

$$\tilde{\Phi} = \int_0^\infty e^{-st}\,\Phi(x,y,t)\,dt,$$

$$s \geq \epsilon_1 > 0,$$

$$\tilde{p}_t = \int_0^\infty e^{-st}\,p_t(x,t)\,dt.$$

It is obtained from (2.1) and (2.2) that

$$\tilde{\Phi}_{xx} + \tilde{\Phi}_{yy} = 0,$$

$$s^2\tilde{\Phi} + g\tilde{\Phi}_y = s\Phi(x,0,0) + \Phi_t(x,0,0) - \frac{1}{\rho}\,\tilde{p}_t,$$

$$\tilde{\Phi}_n = 0.$$

We introduce two new independent variables $x', y'$ such that

$$x' = \frac{s^2}{g}\,x, \qquad y' = \frac{s^2}{g}\,y,$$

and the above equations become

$$\tilde{\phi}_{x'x'} + \tilde{\phi}_{y'y'} = 0 \quad .$$

$$(2.3) \qquad \tilde{\phi} + \tilde{\phi}_{y'} = \tilde{F}(x',s),$$

$$\tilde{\phi}_{n'} = 0 \quad .$$

where

$$\tilde{\phi}(x',y',s) = \tilde{\Phi}(x,y,s),$$

$$\tilde{F}(x',s) = \frac{1}{s}\Phi(x,0,0) + \frac{1}{s^2}\Phi_t(x,0,0) - \frac{1}{\rho s^2}\tilde{p}_t .$$

The whole problem is now reduced to solving the Laplace equation for a sector with inhomogeneous mixed condition on one boundary and zero normal derivative on the other. The method to find a solution for (2.5) is motivated by what has been done for the case of a progressive wave over a sloping beach [5]. Let

$$x' + iy' = r'(\cos\theta + i\sin\theta) .$$

$$\phi(r',\theta,s) = \tilde{\phi}(x',y',s),$$

$$F(r',s) = \tilde{F}(x',s) .$$

Using $r',\theta$ as independent variables, we have from (2.5)

$$(r')^2\phi_{r'r'} + r'\tilde{\phi}_{r'} + \tilde{\phi}_{\theta\theta} = 0,$$

$$(2.4) \qquad r'\phi + \phi_\theta = r'F(r',s), \qquad \theta = 0$$

$$\phi_\theta = 0 . \qquad \theta = -\gamma ,$$

Now we apply Laplace transform to (2.4) with respect to $r'$. Let

$$\phi^* = \int_0^\infty e^{-\tau r'} \tilde{\phi}(r',\theta,s)dr',$$

$$\tau \geq \epsilon_2 > 0,$$

$$F^* = \int_0^\infty e^{-\tau r'} \tilde{F}(r',s)dr',$$

and it is obtained from (2.4) that

$$\tau^2\phi^*_{\tau\tau} + 3\tau\phi^*_\tau + \phi^* + \phi^*_{\theta\theta} = 0,$$

$$\phi^*_\tau - \phi^*_\theta = F^*_\tau , \qquad \theta = 0$$

$$\phi^*_\theta = 0, \qquad \theta = -\gamma .$$

By introducing two new variables $\Psi = \Psi(p,\theta)$ and $p$ such that

$$\phi^* = -e^{p}\Psi ,$$

$$\tau = e^{-p},$$

we obtain from the above equations

$$\Psi_{\rho\rho} + \Psi_{\theta\theta} = 0,$$

$$(2.5) \qquad \Psi_{\rho} - e^{-\rho}\Psi_{\theta} + \Psi = -F_{\rho}e^{-\rho}, \qquad \theta = 0,$$

$$\Psi_{\theta} = 0, \qquad \theta = -\gamma,$$

where

$$F(\rho,s) = F^{*}(\tau,s)^{(1)}$$

Since $\Psi(\rho,\theta)$ is a solution of the two-dimensional Laplace equation defined in a strip, we assume that there exists an analytic function $f(w)$ such that

$$\Psi = \mathrm{Re}\left\{ f(w) \right\},$$

where $w = \rho + i\theta$ and "Re" stands for "the real part of". It follows that

$$\Psi_{\rho} = \mathrm{Re}\left\{ f'(w) \right\}.$$

$$\Psi_{\theta} = \mathrm{Re}\left\{ if'(w) \right\}.$$

---

(1) Sometimes we shall write $F(\rho)$ for $F(\rho,s)$.

From the mixed boundary condition at $\theta = 0$, we have

$$\mathrm{Re}\left\{ f'(w) + f(w) - e^{-w} i f'(w) \right\} = \mathrm{Re}\left\{ -F'(w) e^{-w} \right\}, \qquad \theta = 0.$$

By the principle of reflection, we obtain

$$\begin{aligned} & f'(w) + f(w) + e^{-w} i f'(w) + F'(w) e^{-w} \\ (2.6) \qquad & \\ & = -\bar{f}'(w) - \bar{f}(w) + e^{-w} i \bar{f}'(w) - F'(w) e^{-w} \end{aligned}$$

where we assume that the conjugate of $F(w)$, $\overline{F(w)}$, is equal to $F(\bar{w})$, and $\bar{f}(w)$ denotes the function $f(w)$ with its coefficients replaced by their conjugates. From the boundary condition at $\theta = -\gamma$, we have

$$\mathrm{Re}\left\{ i f'(w) \right\} = 0 .$$

It follows from the principle of reflection that

$$\begin{aligned} & f'(w - i2\gamma) = \bar{f}'(w) , \\ (2.8) \quad & \text{or,} \\ & f(w - i2\gamma) = \bar{f}(w) + c_1 , \end{aligned}$$

where $c_1$ is an arbitrary constant. By combining (2.6) and (2.8), it is found that

$$(2.9)\quad (e^{w}+i)f(w) = -(e^{w}-i)f(w-i2\gamma)-2F(w) + c_1 e^{w} + c_2,$$

where $c_2$ is also an arbitrary constant. Suppose that a solution of $f(w)$ has been found, which satisfies the conditions of the inversion theorem of Laplace transform, then

$$\tilde{\phi} = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\tau r'}\frac{1}{2\tau}[f(w) + \bar{f}(\bar{w})]d\tau$$

$$= -\frac{1}{2\pi i}\int_{P} e^{\tau r'}\frac{1}{2\tau}[f(-\ell n\tau + i\theta) + \bar{f}(-\ell n\tau - i\theta)]d\tau,$$

where we assume that the path of integration of the first integral can be deformed to a contour $P$ symmetric with respect to the real axis in the $\tau$-plane. It is also assumed that the contour integral for $\tilde{\phi}$ remains unchanged as the path $P\exp(-i\theta)$ is rotated back to $P$. Now let

$$\tau = \zeta e^{i\theta}, \qquad f(-\ell n\zeta) = h(\zeta) .$$

we have

$$(2.10)\qquad \tilde{\phi} = -\mathrm{Re}\,\frac{1}{\partial\pi i}\int_{P} e^{\frac{s^2 z}{g}\zeta}\,\frac{1}{\zeta}\,h(\zeta)d\zeta,$$

where $z = r\exp(i\theta)$ . In (2.9), we replace $w$ by $-\ell n\zeta$ and $f(-\ell n\zeta)$ by $h(\zeta)$, and obtain

$$(\zeta-i)h(\zeta) = (\zeta+i)h(\zeta e^{i2\gamma}) + 2i\zeta F^*(\zeta) + ic_1 + ic_2\zeta .$$

We may set $c_1 = c_2 = 0$, and only consider the $\zeta$-difference equation

$$(2.11) \quad (\zeta-i)h(\zeta) = (\zeta + i)h(\zeta e^{i2\gamma}) + 2iF^*(\zeta)$$

subject to the condition

$$(2.12) \qquad \mathrm{Im}\{ h(\zeta) \} = 0 , \text{ for } \arg \zeta = \gamma,$$

which is equivalent to

$$\mathrm{Re}\{ if'(w) \} = 0, \quad \text{for } \theta = -\gamma,$$

if $c_1 = 0$. It is shown that if an analytic function $h(\zeta) = f(w)$ satisfying (2.11) and (2.12) can be found, $\Psi = \mathrm{Re}\{ f(w) \}$ will meet (2.5) and $\tilde{\phi}$ given by (2.10) is then a solution of (2.5). An indirect but equivalent method for the derivation of (2.10) and (2.11) will be given in Appendix I. In the approach there, we assume that both $\tilde{F}(x')$ and $\tilde{\phi}$ have a Laplace integral representation. Therefore, we can avoid the Laplace transform with respect to $r'$ and the requirement that

$$\frac{h(\zeta)}{\zeta} \longrightarrow 0, \quad \text{as } \zeta \longrightarrow \infty,$$

for the inversion of the Laplace transform can be dropped. Instead, we must impose the boundedness condition at the origin in order to determine the solution uniquely.

## 3. Solution of the q-difference equation.

In this section we are going to construct a solution of (2.11) subject to the condition (2.12). Let

$$h(\zeta) = h_1(\zeta)h_2(\zeta),$$

where $h_1(\zeta)$ satisfies

$$(3.1) \qquad (\zeta-i)h_1(\zeta) = (\zeta+i)h_1(\zeta e^{i2\gamma}) \quad .$$

From (2.11), we see that $h_2(\zeta)$ satisfies

$$(3.2) \qquad h_2(\zeta e^{i2\gamma}) = h_2(\zeta) - \frac{2i\zeta F^*(\zeta)}{(\zeta-i)h_1(\zeta)} \quad .$$

The solution of (3.1) which meets (2.12) can be obtained from the solution for $\frac{1}{h_1(\zeta)}$, satisfying

$$(\zeta-i)h_1^{-1}(\zeta e^{i2\gamma}) = (\zeta+i)h_1^{-1}(\zeta).$$

The above equation is the same as the basic functional relation found in [2], and the solution is

$$h_1^{-1}(\zeta) = \frac{\zeta}{\zeta+i} \exp\left\{ -\frac{1}{\pi}\int_0^\infty \ln\left(\frac{v^{\pi/\gamma}-1}{v^{\pi/\gamma}}\frac{v^2}{v^2-1}\right)\frac{\zeta}{v^2+\zeta^2}\,d\zeta\right\},$$

for $-\pi/2 \leq \arg\zeta \leq \pi/2$. As $\zeta \longrightarrow \infty$, $h_1^{-1}(\zeta) \longrightarrow 1$ and as $\zeta \longrightarrow 0$, $h_1^{-1}(\zeta) \longrightarrow -i\zeta^{\pi/2\gamma}$. Furthermore, $h_1^{-1}(\zeta)$ is analytic in $-\pi/2 < \arg\zeta < \pi/2 + 2\gamma$, and has simple poles at $\zeta = -i$, $\pm\, ie^{\pm i2n\gamma}$, $n = 1,2,\ldots$ . Therefore,

$$(3.3)\quad h_1(\zeta) = \frac{\zeta+i}{\zeta} \exp\left\{ \frac{1}{\pi}\int_0^\infty \ln\left(\frac{v^{\pi/\gamma}-1}{v^{\pi/\gamma}}\frac{v^2}{v^2-1}\right)\frac{\zeta}{v^2+\zeta^2}\,dv\right\},$$

and the behavior of $h_1(\zeta)$ can be deduced directly from that of $h_1^{-1}(\zeta)$. Since $h_1^{-1}(\zeta)$ satisfies the condition $\mathrm{Re}\left\{ h_1^{-1}(\zeta)\right\} = 0$ at $\arg\zeta = \gamma$, so does $h_1(\zeta)$. If we let

$$H_1(\zeta) = \frac{\zeta-i}{\zeta}\, h_1(\zeta),$$

then from (3.1),

$$(\zeta+i)H_1(\zeta e^{i2\gamma}) = (\zeta-ie^{-i2\gamma})H_1(\zeta).$$

It is easily shown [5] that as $\gamma = \frac{\pi}{2m}$, $m$ = positive integers,

$$(3.4)\qquad H_1(\zeta) = \frac{\prod_{q=0}^{m}(\zeta - ie^{i2q\gamma})}{\zeta^m},$$

which is analytic and single-valued in the whole $\zeta$-plane

except having a pole at $\zeta = 0$.

Since $h_1(\zeta)$ has been found, it is not difficult to find a solution for $h_2(\zeta)$. In Appendix II, we shall show that the following integral representation, for $0 \leq \arg \zeta \leq 2\gamma$,

$$(2.5) \qquad h_{2p}(\zeta) = \frac{1}{\pi}\int_0^\infty \frac{\xi^{\gamma/\pi} F^*(\xi^{\gamma/\pi})\,d\xi}{(\zeta^{\gamma/\pi}-i)h_1(\xi^{\gamma/\pi})(\zeta-\xi^{\pi/\gamma})},$$

furnishes a particular solution for $h_2(\zeta)$, which also satisfies (2.12). The analytical continuation of $h_{2p}(\zeta)$ beyond the sector can be achieved by shifting the path of integration or using (2.11). The general solution for $h_2(\zeta)$ is

$$h_2(\zeta) = h_{2p}(\zeta) + h_o(\zeta),$$

where $h_o(\zeta)$ is a solution of

$$h_o(\zeta) = h_o(\zeta e^{i2\gamma}) .$$

As shown in [5], $h_o(\zeta)$ has the only solution

$$h_o(\zeta) = \mu_p \zeta^{p\pi/\gamma} , \quad p = 0, \pm 1, \pm 2, \ldots,$$

where $\mu_p$ are real constants. However, by the requirement of the inversion theorem of Laplace transform, $\frac{h(\zeta)}{\zeta}$ must tend to zero as $\zeta \longrightarrow \infty$ and consequently we must set $\mu_p = 0$ for $p = 1,2,\ldots$. From (2.10), we have

$$(3.6)\quad \tilde{\phi} = -\mathrm{Re}\,\frac{1}{2\pi i}\int_P e^{\frac{s^2 z}{g}\zeta}\,\frac{1}{\zeta}\,h_1(\zeta)(h_{2p}(\zeta) + h_o(\zeta))d\zeta\ .$$

The constants $\mu_p$ for $p = 0, -1, -2, \ldots$ can be determined by the boundedness condition at infinity. Suppose that $\tilde{\phi}$ satisfies the condition for the inversion theorem of Laplace transform, then

$$\Phi = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st}\,\tilde{\phi}\,ds, \qquad c > 0\ .$$

To see how $\mu_p$ are determined, we must first investigate the singularities of $h_2(\zeta)$ and its behavior near $\zeta = 0$. For simplicity, let us assume that $F^*(\zeta)$ has simple poles only in $\mathrm{Re}\zeta < 0$, say at $\zeta = \zeta_+$ in the upper half-plane and $\zeta = \zeta_-$ in the lower half-plane, and $F^*(0) \neq 0$. From (3.2), we see that $h_2(\zeta)$ has simple poles at $\zeta = \zeta_+ e^{i2n\gamma}$ $n = 1,2,\ldots$. We may replace $\zeta$ by $\zeta e^{-i2\gamma}$ in (3.2) and obtain

$$(3.7)\qquad h_2(\zeta e^{-i2\gamma}) = h_2(\zeta) + \frac{2i\zeta e^{-i2\gamma}\,F^*(\zeta e^{-i2\gamma})}{(\zeta e^{-i2\gamma}-i)h_1(\zeta e^{-i2\gamma})}\ .$$

It shows that $h_2(\zeta)$ has simple poles at $\zeta = \zeta_- e^{-i2n\gamma}$, $n = 0,1,\ldots$. We already know that $h_1(\zeta)$ has simple zeros at $\zeta = -i, \pm ie^{\pm i2n\gamma}$, $n = 1,2,\ldots$. Then from (3.2) and (3.7) $h_2(\zeta)$ has simple poles at $\zeta = -i, \pm e^{\pm i2n\gamma}$, $n = 1,2,\ldots$. Hence the poles of $h(\zeta) = h_1(\zeta)h_2(\zeta)$ in the finite slit $\zeta$ - plane only come from the contribution of $F^*(\zeta)$. As $\zeta \longrightarrow 0$, it is found from the solution for $h_2(\zeta)$ that

$$h_{2p}(\zeta) = d + O(\zeta^{\pi/2\gamma+1}) \qquad 0 < \gamma < \pi/2,$$

$$= d + O(\zeta^{\pi/2\gamma+1}\ln\zeta) \qquad \gamma = \pi/2,$$

$$= d + O(\zeta^{\pi/\gamma}) \qquad \pi/2 < \gamma \leq \pi .$$

where

$$d = \frac{1}{\pi}\int_0^\infty \frac{\xi^{\gamma/\pi-1}F^*(\xi^{\gamma/\pi})d\xi}{(\xi^{\gamma/\pi}-i)h_1(\xi^{\gamma/\pi})} = \text{real}.$$

Assuming that $z$ is sufficiently large, we deform the path $P$ in (2.6) to a path $D$ as shown in Fig. 2. As $z \longrightarrow \infty$, the contribution due to the poles of $h_{2p}(\zeta)$ vanishes, and the behavior of $\tilde{\phi}$ at infinity depends upon the behavior of $h_1(\zeta)(h_{2p}(\zeta) + h_0(\zeta))$ at $\zeta = 0$. We see that, as $z \longrightarrow \infty$,

$$\frac{1}{2\pi i}\int_P e^{\frac{s^2 z}{g}\zeta}\frac{1}{\zeta}h_1(\zeta)h_{2p}(\zeta)d\zeta$$

$$= \frac{d}{2\pi i}\int_D e^{\frac{s^2 z}{g}\zeta}\frac{1}{\zeta}h_1(\zeta)d\zeta + O(1) \sim \frac{d}{\Gamma[1+\frac{\pi}{2\gamma}]}\left(\frac{s^2 z}{g}\right)^{\pi/2\gamma}$$

and

$$\frac{1}{2\pi i}\int_P e^{\frac{s^2 z}{g}}\frac{1}{\zeta}h_1(\zeta)h_0(\zeta)d\zeta$$

$$\sim \frac{\mu_o}{2\pi i}\int_D e^{\frac{s^2 z}{g}\zeta}\frac{1}{\zeta}h_1(\zeta)d\zeta + \sum_{p<0}\frac{\mu_p}{\pi[1-(p-\frac{1}{2})\frac{\pi}{\gamma}]}\left(\frac{s^2 z}{g}\right)^{-(p-\frac{1}{2})\frac{\pi}{\gamma}} .$$

By the boundedness condition at infinity, we have

$$\mu_p = 0 \quad \text{for} \quad p = -1,-2,\ldots \quad \text{and}$$

$$\mu_0 = -d .$$

The solution for $\tilde{\phi}$ is given by

$$(5.8) \qquad \tilde{\phi} = -\,\mathrm{Re}\left\{\frac{1}{2\pi i}\int_P e^{\frac{s^2 z}{g}\zeta}\frac{1}{\zeta}h_1(\zeta)[h_{2p}(\zeta)-d]d\zeta\right\}.$$

The integral given above may not converge for the whole range $0 < \mathrm{Re}\ z < \infty$ in the sector. For instance, if we shift the function $\tilde{f}(r',s)$ to the right by a distance $a$ and let $\tilde{f}(r',s) = 0$ for $r' < a$. Then $F^*(\zeta)$ is replaced by $e^{-a\zeta}F^*(\zeta)$. Therefore, for a conveniently chosen path $P$, the integral in (5.8) may only converge for $\mathrm{Re}\ z > a$. However, the solution for $\tilde{\phi}$ in $\mathrm{Re}\ z < a$ can be obtained by analytic continuation as shown by an exam le in the next section.

## 4. Transient problem of an oscillatory point pressure acting on the equilibrium surface of a 45° beach

As remarked before, for $\gamma = \pi/2m$, $m$ = positive integers

$$H_1(\zeta) = \frac{\prod_{q=0}^{m}(\zeta - ie^{i2q\gamma})}{\zeta^{m+1}} ,$$

which is analytic and single-valued in the whole $\zeta$-plane except that it has a pole at the origin. Because of the simple form of $H_1(\zeta)$ as $\gamma = \frac{\pi}{2m}$, an explicit solution of the problem can be obtained and the manipulations are greatly simplified. We only consider below the case $m = 2$ as an illustration. Nevertheless, the method used here can also be applied to the case of any $m$.

Let

$$p_t = \delta(x-a) \sin wt,$$

which corresponds to an oscillatory point pressure applied at a distance $x = a$ from the shore on the surface $y = 0$. Furthermore, we assume that

$$\phi(0,x,0) = \phi_t(0,x,0) = 0 .$$

Then

$$\tilde{F}(x',s) = -\frac{1}{\rho s^2}\frac{w}{s^2+w^2}\,\delta\left(\frac{gx'}{s^2} - a\right)$$

$$= -\frac{1}{\rho g}\frac{w}{s^2+w^2}\,\delta\left(x' - \frac{s^2 a}{g}\right),$$

where

$$\delta\left(\frac{g}{s^2}\left(x'-\frac{s^2a}{g}\right)\right) = \frac{s^2}{g}\delta\left(x'-\frac{s^2a}{g}\right) .$$

It follows that

$$F^*(\zeta,s) = -\frac{1}{\rho g}\frac{w}{s^2+w^2}\, e^{-\frac{s^2a\zeta}{g}} .$$

The functional relation for $h_2(\zeta)$ is

$$(4.1) \qquad h_2(\zeta e^{i2\gamma}) = h_2(\zeta) - \frac{2i\zeta F^*(\zeta)}{(\zeta-i)h_1(\zeta)} ,$$

where

$$h_1(\zeta) = \frac{\zeta}{\zeta-i}H_1(\zeta) = \frac{(\zeta+1)(\zeta+i)}{\zeta^2} ,$$

and $\tilde{\phi}$ is given by

$$\tilde{\phi} = -\mathrm{Re}\,\frac{1}{2\pi i}\int_P e^{\frac{s^2z}{g}\zeta}\,\frac{h_1(\zeta)}{\zeta}\,(h_{2p}(\zeta)+\mu_0)d\zeta ,$$

where

$$\mu_0 = -\frac{1}{\pi}\int_0^\infty \frac{\zeta^{\gamma/\pi-1}F^*(\zeta^{\gamma/\pi})d\zeta}{(\zeta^{\gamma/\pi}-i)h_1(\zeta^{\gamma/\pi})} .$$

Since $F^*(\zeta,s)$ is an entire function of $\zeta$, $h_{2p}(\zeta)$ is analytic in the finite $\zeta$-plane with a slit along the negative real

$\zeta$-axis. Assuming that $x > a$, we may deform the path $P$ to coincide with the two edges of the slit, and obtain

$$(4.2)\quad \tilde{\phi} = -\mathrm{Re}\,\frac{1}{2\pi i}\int_0^\infty e^{-\frac{s^2 z}{g}r}\,\frac{h_1(re^{i\pi})}{r}\,[h_{2p}(re^{i\pi})-h_{2p}(re^{-i\pi})]\,dr,$$

where by (4.1),

$$h_{2p}(re^{i\pi}) - h_2(re^{-i\pi}) = \frac{w}{s^2+w^2}\,\frac{2i\rho^3}{\rho g}\,\Big[\,e^{-\frac{as^2 r}{g}}\,\frac{1}{(r^2+1)(r+1)}$$

$$(4.3)\quad + e^{\frac{as^2 r}{g}}\,\frac{1}{(r^2+1)(r-1)} + e^{-i\frac{as^2 r}{g}}\,\frac{1}{(r^2-1)(r-i)}$$

$$+ e^{i\frac{as^2}{g}r}\,\frac{1}{(r^2-1)(r+i)}\,\Big]\,.$$

Substituting (4.3) in (4.2), we have

$$\tilde{\phi} = -\mathrm{Re}\,\frac{1}{\pi}\,\frac{w}{s^2+w^2}\Big\{\int_0^\infty \frac{dr}{r+1}\,\Big[(1+i)e^{-\frac{s^2}{g}(z+a)r} + e^{-\frac{s^2}{g}(z+ia)r}$$

$$+ ie^{-\frac{s^2}{g}(z-ia)r} + \int_0^\infty \frac{dr}{r+i}\Big[-ie^{-\frac{s^2}{g}(z+a)r} + e^{-\frac{s^2}{g}(z-ia)r}\Big]\Big\}\,.$$

$$+ (1-i)e^{-\frac{s^2}{g}(z-ia)r}\Big]\Big\}.$$

It is seen that the only term in the above expression, which diverges for $\mathrm{Re}\, z < a$ is

$$I(z) = \int_0^\infty e^{-\frac{s^2}{g}(z-a)r} \frac{dr}{r+i} ,$$

However, let

$$\eta = (r+i)\frac{s^2}{g}(z-a) .$$

and

$$I(z) = e^{i\frac{s^2}{g}(z-a)} \int_{i\frac{s^2}{g}(z-a)}^{i\frac{s^2}{g}(z-a) + \infty \arg(z-a)} \frac{1}{\eta} e^{-\eta} d\eta .$$

The path of integration for $I(z)$ in the slit z-plane can be deformed to the path $\Gamma$ shown in Fig. 3, and

$$I(z) = e^{i\frac{s^2}{g}(z-a)} \int_{i\frac{s^2}{g}(z-a)}^{\infty} \frac{1}{\eta} e^{-\eta} d\eta$$

$$= -e^{i\frac{s^2}{g}(z-a)} Ei(-i\frac{s^2}{g}(z-a)), \text{ for } \operatorname{Re} z > a.$$

The above expression, however, also yields the analytic continuation of $I(z)$ for $\operatorname{Re} z < a$ in the slit z-plane. Now we can express $\tilde{\phi}$ in terms of exponential integrals,

$$\tilde{\phi} = \operatorname{Re} \frac{1}{\pi} \frac{w}{s^2+w^2} [(1+i)e^{-\frac{s^2}{g}(z+a)} Ei(-\frac{s^2}{g}(z+a))$$

$$+ e^{\frac{s^2}{g}(z+ia)} Ei(-\frac{s^2}{g}(z+ia)) + ie^{\frac{s^2}{g}(z-ia)} Ei(-\frac{s^2}{g}(z-ia))$$

$$- ie^{\frac{s^2}{g}(z+a)} Ei(-\frac{s^2}{g}i(z+a) + e^{\frac{s^2}{g}(z-a)} Ei(-\frac{s^2}{g}i(z-a))$$

$$+ (1-i)e^{-\frac{s^2}{g}(z-ia)} Ei(-\frac{s^2}{g}i(z-ia))] ,$$

and making use of the transform pair [11],

$$\int_0^\infty e^{-st}[2i\pi^{1/2}\alpha e^{-\alpha^2 t^2}\ \mathrm{Erf}(i\alpha t)]dt = e^{\frac{s^2}{4\alpha^2}}\ \mathrm{Ei}(-\tfrac{1}{4}\alpha^{-2}s^2),$$

$$|\arg\alpha| < \pi/4$$

we have, for $y < 0$,

$$\phi = \mathrm{Re}\ \frac{i}{\sqrt{\pi g}}\rho\ \sin wt\ *\ \Bigg\{\frac{(1+i)}{\sqrt{z+a}}\exp\left(-\frac{gt^2}{4(z+a)}\right)\mathrm{Erf}\left(\frac{i}{2}\frac{\sqrt{gt}}{\sqrt{z+a}}\right)$$

$$+\frac{1}{\sqrt{z+ia}}\exp\left(-\frac{gt^2}{4(z+ia)}\right)\mathrm{Erf}\left(\frac{i}{2}\frac{\sqrt{gt}}{\sqrt{z+ai}}\right)$$

$$+\frac{i}{\sqrt{z-ia}}\exp\left(-\frac{gt^2}{4(z-ia)}\right)\mathrm{Erf}\left(\frac{i}{2}\frac{\sqrt{gt}}{\sqrt{z-ia}}\right)$$

$$-\frac{i}{\sqrt{i(z+a)}}\exp\left(-\frac{gt^2}{4i(z+a)}\right)\mathrm{Erf}\left(\frac{i}{2}\frac{\sqrt{gt}}{\sqrt{i(z+a)}}\right)$$

$$+\frac{1}{\sqrt{i(z-a)}}\exp\left(-\frac{gt^2}{4i(z-a)}\right)\mathrm{Erf}\left(\frac{i}{2}\frac{\sqrt{gt}}{\sqrt{i(z-a)}}\right)$$

$$+\frac{(1-i)}{\sqrt{i(z-ia)}}\exp\left(-\frac{gt^2}{4i(z-ia)}\right)\mathrm{Erf}\left(\frac{i}{2}\frac{\sqrt{gt}}{\sqrt{i(z-ia)}}\right)\Bigg\}.$$

where "*" stands for the convolution with respect to $t$.

Let us now consider the limiting case $t \longrightarrow \infty$. Here we make use of the following integrals [12]:

$$i\int_0^\infty e^{-\alpha^2 t^2}\,\mathrm{Erf}(i\alpha t)\,\mathrm{Cos}\,wt\,dt = \frac{1}{2}\pi^{-i/2}\,\alpha^{-1}\,e^{\frac{w^2}{4\alpha}}\left(\mathrm{Ei}\left(\frac{w^2}{4\alpha^2}\right) + i\pi\right),$$

$$\int_0^\infty e^{-\alpha^2 t^2}\,\mathrm{Erf}(i\alpha t)\,\sin wt\,dt = i\,\frac{1}{2}\pi^{1/2}\,\alpha^{-1}e^{-\frac{w}{4\alpha^2}},\quad |\arg\alpha|<\pi/4,$$

and the steady-state solution is found to be

$$\phi = \mathrm{Re}\,\frac{\sin wt}{\pi\rho g}\Big\{(1+i)\exp\left(-\frac{w^2(z+a)}{g}\right)\mathrm{Ei}\left(\frac{w^2(z+a)}{g}\right)$$

$$+\exp\left(-\frac{w^2(z+ia)}{g}\right)\mathrm{Ei}\left(\frac{w^2(z+ia)}{g}\right) + i\exp\left(-\frac{w^2(z-ia)}{g}\right)$$

$$\mathrm{Ei}\left(\frac{w^2(z+ia)}{g}\right)$$

(4.4)

$$-i\exp\left(-\frac{iw^2(z+a)}{g}\right.\mathrm{Ei}\left(\frac{iw^2(z+a)}{g}\right) + \exp\left(-\frac{iw^2(z-a)}{g}\right.\mathrm{Ei}\left(\frac{iw^2(z-a)}{g}\right.$$

$$+(1-i)\exp\left(-\frac{iw^2(z-ia)}{g}\right.\mathrm{Ei}\left(\frac{iw^2(z-ia)}{g}\right)\Big\} + \mathrm{Re}\,\frac{1}{\rho g}\,e^{iwt}$$

$$\Big\{(1+i)\exp\left(-\frac{w^2(z+a)}{g}\right) + \exp\left(-\frac{w^2(z+ia)}{g}\right) + i\exp\left(-\frac{w^2(z-ia)}{g}\right)$$

$$-i\exp\left(-\frac{iw^2(z+a)}{g}\right. + \exp\left(-\frac{iw^2(z-a)}{g}\right) + (1-i)\exp\left(-\frac{iw^2(z-ia)}{g}\right)\Big\}.$$

The expression in the second brace of the above solution can be written in the form

$$\mathrm{Re}\left\{ e^{iwt}\,\frac{1}{g}\left(e^{-\frac{w^2a}{g}} + \cos\frac{w^2a}{g} - \sin\frac{w^2a}{g}\right)\left[(1+i)e^{-\frac{w^2z}{g}} + (1-i)e^{-i\frac{w^2z}{g}}\right]\right\}.$$

In the following, we investigate the solution at $z = \infty$, 0, and $a$. First, if we let $z \longrightarrow \infty$, we see that

$$(4.5)\quad \phi \longrightarrow \mathrm{Re}\,\frac{(1-i)}{g}\left(e^{\frac{w^2a}{g}} + \cos\frac{w^2a}{g} - \sin\frac{w^2a}{g}\right)e^{i(wt-\frac{w^2z}{g})}$$

$$= \sqrt{\frac{2}{g}}\left(e^{-\frac{w^2a}{g}} + \cos\frac{w^2a}{g} - \sin\frac{w^2a}{g}\right)\cos\left(wt - \frac{w^2z}{g} - \frac{\pi}{4}\right).$$

There is the so-called radiation condition for a steady-state solution. As $z \longrightarrow 0$, we have

$$\phi \longrightarrow \frac{2}{\pi g}\sin wt\left\{ e^{-\frac{w^2a}{g}}\,\mathrm{Ei}\left(\frac{w^2a}{g}\right) + \mathrm{Re}\sqrt{2}\; e^{i(\frac{w^2a}{g} + \frac{\pi}{4})}\,\mathrm{Ei}\left(-\frac{iw^2a}{g}\right)\right\} + \frac{2}{g}\cos wt\left(e^{-\frac{w^2a}{g}} + \cos\frac{w^2a}{g} - \sin\frac{w^2a}{g}\right)$$

$$= A\cos(wt - \theta),$$

where

$$A = \frac{2}{g}\left[B_1^2\ 1\ B_2^2\right]^{1/2}$$

$$\theta = \arctan \frac{B_1}{B_2} ,$$

$$B_1 = e^{-\frac{w^2 a}{g}} Ei\left( \frac{w^2 a}{g} \right) + \sqrt{2} \cos\left( \frac{w^2 a}{g} + \frac{\pi}{4} \right) Ci\left(\frac{w^2 a}{g}\right) +$$

$$\sqrt{2} \sin\left(\frac{w^2 a}{g} + \frac{\pi}{4}\right) Si\left( \frac{w^2 a}{g} \right),$$

$$B_2 = \pi\left(e^{-\frac{w^2 a}{g}} + \cos \frac{w^2 a}{g} - \sin \frac{w^2 a}{g}\right) .$$

Near $z = a - io$ it is easily seen from (3.4) that

$$\phi \longrightarrow \frac{1}{\pi g \rho} \sin wt \log\left( \frac{w^2}{g} |z - a| \right) .$$

5. <u>Steady-state problem of an oscillatory point pressure acting on the equilibrium surface of a sloping beach.</u>

Let

$$p_t = \delta(x-a) \sin wt.$$

We assume that a particular solution of $\Phi$ takes the form

$$\Phi_p = \phi(x,y) \sin wt.$$

Substitution of $\Phi_p$ in (2.1) yields

$$\phi_{xx} + \phi_{yy} = 0, \qquad 0 < r < \infty, \quad -\gamma < \theta < 0,$$

$$-w^2\phi + g\phi_y = -\frac{1}{\rho}\delta(x-a) \qquad 0 < r < \infty, \qquad \theta = 0,$$

$$\phi_r = 0, \qquad 0 < r < \infty, \quad \theta = -\gamma.$$

Let us introduce two new independent variables $x^*$, $y^*$ such that

$$x^* = \frac{w^2}{g}x, \qquad y^* = \frac{w^2}{g}y .$$

In terms of $x^*$, $y^*$, the above equations become

$$\tilde{\phi}_{x^*x^*} + \tilde{\phi}_{y^*y^*} = 0 ,$$

$$\tilde{\phi} - \tilde{\phi}_{y^*} = \frac{1}{\rho g}\delta(x^* - \frac{w^2 a}{g}),$$

$$\tilde{\phi}_{\eta^*} = 0 .$$

We follow exactly what we did in Sec.2 and Sec.3 and obtain

$$(5.1) \qquad \phi = -\mathrm{Re}\,\frac{1}{2\pi i}\int_P e^{\frac{w^2 z}{g}\zeta}\,\frac{1}{\zeta}\,g(\zeta)d\zeta$$

where $g(\zeta)$ satisfies

$$(5.2)\quad (\zeta+i)g(\zeta) = (\zeta-i)g(\zeta e^{i2\gamma}) - \frac{2i\zeta}{\rho g}\, e^{-\frac{w^2 a\zeta}{g}} ,$$

$$(5.3)\quad \operatorname{Im} g(\zeta e^{i\gamma}) = 0, \quad \text{for } \zeta \text{ real, positive.}$$

Let

$$g(\zeta) = g_1(\zeta)g_2(\zeta),$$

and

$$(\zeta+i)g_1(\zeta) = (\zeta-i)g_1(\zeta e^{i2\gamma}) .$$

From (5.2), we have

$$(5.4)\quad g_2(\zeta) = g_2(\zeta e^{i2\gamma}) - \frac{2i\zeta e^{-\frac{w^2 a\zeta}{g}}}{\rho g(\zeta+i)g_1(\zeta)}$$

Since $g_1(\zeta)$ satisfies the same equations as $h_1^{-1}(\zeta)$, the solution for $g_1(\zeta)$ is, [2]

$$(5.5)\quad g_1(\zeta) = \frac{\zeta}{\zeta+i} \exp\left\{ -\frac{1}{\pi}\int_0^\infty \ln\left(\frac{v^{\pi/\gamma}-1}{v^{\pi/\gamma}}\,\frac{v^2}{v^2-1}\right)\frac{\zeta}{v^2+\zeta^2}\, d\zeta \right\},$$

$$\pi/2 \leq \arg \zeta \leq \pi/2 .$$

As $\zeta \longrightarrow \infty$, $g_1(\zeta) \longrightarrow 1$, and as $\zeta \longrightarrow 0$, $g_1(\zeta) \longrightarrow \zeta^{\pi/2\gamma}$. Furthermore, $g_1(\zeta)$ is analytic in $\mathrm{Re}(\zeta) > 1$ and has simple poles at $\zeta = -i$, $ie^{\pm 2n\gamma} = \zeta_n$, $n = 1,2,\ldots$ . A particular solution $g_{2p}(\zeta)$ of $g_2(\zeta)$ can be found by the same method as we used before. From (5.4), we have

$$(5.6) \qquad g_{2p}(\zeta) = -\frac{1}{\pi\rho g}\int_0^\infty \frac{\xi^{\gamma/\pi}\exp\left(-\frac{w^2 a}{g}\xi^{\gamma/\pi}\right)d\xi}{(\xi^{\gamma/\pi}+i)g_1(\xi^{\gamma/\pi})(\xi-\zeta^{\pi/\gamma})},$$

$$0 \leq \arg\zeta \leq 2\gamma,$$

which also meets the condition (5.3) as can be easily verified. An analytical continuation of $g_{2p}(\zeta)$ beyond the sector $0 \leq \arg\zeta \leq 2\gamma$ can be obtained either by (5.4) or by shifting the path of integration of (5.6). It is then obtained from (5.1) that

$$\phi = -\mathrm{Re}\frac{1}{2\pi i}\int_P e^{\frac{w^2 z}{g}\zeta}\,\frac{1}{\zeta}g_1(\zeta)\,g_{2p}(\zeta)d\zeta\ .$$

In the following, we show that $\phi$ is bounded at infinity. We deform the contour $P$ to a contour $D$ as shown in Fig. 2, and

$$\phi = -\mathrm{Re}\,\frac{1}{2\pi i}\int_D e^{\frac{w^2 z}{g}\zeta}\,\frac{1}{\zeta}\,g_1(\zeta)\,g_{2p}(\zeta)d\zeta$$

$$-\mathrm{Re}\sum_n e^{\frac{w^2 z}{g}}\,g_{2p}(\zeta_n)\,\frac{1}{\zeta_n}[(\zeta-\zeta_n)g_1(\zeta)]_{\zeta=\zeta_n}$$

$$-\mathrm{Re}\ ie^{-\frac{iw^2 z}{g}}\ g_{2p}(-i)[(\zeta + i)g_1(\zeta)]_{\zeta = -i} \ .$$

We can easily see that as $z \longrightarrow \infty$

$$\phi \longrightarrow -\ \mathrm{Re}\ ie^{\frac{iw^2 z}{g}}\ g_{2p}(-i)[(\zeta + i)g_1(\zeta)]_{\zeta = -i} \ .$$

In order to find a complete steady-state solution of (2.1), we must consider the steady-state solutions of the homogeneous equations obtained from (2.1) with $\equiv 0$ . It has been found in [2] that the only solution of

$$\bar{\phi}_{xx} + \bar{\phi}_{yy} = 0,$$

$$-w^2\bar{\phi} + g\bar{\phi}_y = 0,$$

$$\bar{\phi}_\eta = 0 \ ,$$

which is bounded at infinity and meets the requirement that $\frac{g_1(\zeta)}{\zeta} \longrightarrow 0$ as $\zeta \longrightarrow \infty$ takes the form

$$\bar{\phi} = -\ \mathrm{Re}\ \frac{\lambda}{2\pi i}\int_P e^{\frac{w^2 z}{g}\zeta}\ \frac{1}{\zeta}\, g_1(\zeta)d\zeta \ ,$$

where $\lambda$ is a real constant. As $z \longrightarrow \infty$,

$$\bar{\phi} \longrightarrow -\ \mathrm{Re}\ i\lambda e^{-\frac{iw^2 z}{g}}\ [(\zeta + i)g_1(\zeta)]_{\zeta = -i} \ .$$

Therefore, the complete solution of our problem will be

$$(5.7)\quad \Phi = -\mathrm{Re}\left\{ \frac{\sin wt}{2\pi i} \int_p e^{\frac{w^2 z}{g}\zeta} \frac{1}{\zeta} g_1(\zeta)[g_{2p}(\zeta) + \lambda_1]d\zeta \right.$$

$$\left. + \frac{\cos wt}{2\pi i} \int_p e^{\frac{w^2 z}{g}\zeta} \frac{\lambda_2}{\zeta} g_1(\zeta)d\zeta \right\},$$

where $\lambda_1$, $\lambda_2$ are two real constants. Here to determine $\lambda_1$ and $\lambda_2$, we prescribe the so-called radiation condition at infinity.

As $z \longrightarrow \infty$,

$$\Phi \longrightarrow \mathrm{Re}\ C\ e^{i(wt - \frac{w^2}{g} z)},$$

where $C$ is a complex constant. From the results we obtained before, we find that as $z \longrightarrow \infty$,

$$(5.8)\quad \Phi \longrightarrow -\mathrm{Re}\left\{ ie^{-i\frac{w^2 z}{g}} [(\zeta + i)g_1(\zeta)]_{\zeta = -i} \right.$$

$$\left. [(g_{2p}(-i) + \lambda_1)\sin wt + \lambda_2 \cos wt \right\}.$$

Since $\lambda_1$ and $\lambda_2$ are real, by the radiation condition, we have

$$\lambda_1 = -\mathrm{Re}\ g_{2p}(-i),$$

$$\lambda_2 = \mathrm{Im}\ g_{2p}(-i).$$

By introducing a new variable, $\eta = \zeta^{\gamma/\pi}$, we obtain from (5.6) that

$$g_{2p}(\zeta) = \frac{1}{\gamma\rho g}\int_0^\infty \frac{\eta^{\pi/\gamma}\exp(-\frac{w^2 a\eta}{g})d\eta}{I_1(\eta)(\eta^{\pi/\gamma} - \zeta^{\pi/\gamma})}, \quad 0 \leq \arg\zeta \leq 2\gamma ,$$

where $$I_1(\eta) = (\eta + i)\, g_1(\eta) .$$

Now we may shift the path of integration for $g_{2p}(\zeta)$ to $\arg\zeta = -\pi/2$, and obtain

$$g_{2p}(\zeta) = -\frac{1}{\gamma\rho g}\int_0^\infty \frac{\eta^{\pi/\gamma}\exp(-\frac{w^2 a}{g}\zeta)d\eta}{I_1(\eta)(\eta^{\pi/\gamma} - \zeta^{\pi/\gamma})} ,$$

which defines an analytic function for $-\pi/2 \leq \arg\zeta \leq -\pi/2+2\gamma$, and consequently, as shown in Appendix II, is the analytic continuation of $g_{2p}(\zeta)$ in $-\pi/2 \leq \arg\zeta \leq -\pi/2 + 2\gamma$. The above expression can be rewritten as

$$g_{2p}(\zeta) = \frac{i}{\pi\rho g}\int_0^\infty \frac{\eta^{\gamma/\pi}\exp(-\frac{iw^2 a}{g}\eta^{\gamma/\pi})d\eta}{I(-i\eta^{\gamma/\pi})[\eta-\zeta^{\pi/\gamma}\exp(i\pi^2/2\gamma)]} ,$$

where from (5.5) and Plemlj formula, for $\eta$ real,

$$(5.8) \quad I(-i\eta) = -i\eta\exp\left\{-\frac{1}{2}\ln\left(\frac{\eta-1}{\eta}\,\frac{\eta^{2\gamma/\pi}}{\eta^{2\gamma/\pi}-1}\right) - \frac{1}{\pi i}\fint_0^\infty \ln\left(\frac{v^{\pi/\gamma}-1}{v^{\pi/\gamma}}\,\frac{v^2}{v^2-1}\right)\frac{\eta^{\gamma/\pi}dv}{v^2-\eta^{2\gamma/\pi}}\right\},$$

where "$\fint$" stands for the principal value of the integral. Hence

$$g_{2p}(-i) = -\frac{i}{\rho g}\exp\left(\frac{iw^2 a}{g}\right)\frac{1}{I(-i)} + \frac{i}{\pi\rho g}\fint_0^\infty \frac{\eta^{\gamma/\pi}\exp(\frac{iw^2 a}{g}\eta^{\gamma/\pi})d\eta}{I_1(-i\eta^{\gamma/\pi})(\eta-1)} .$$

and

$$\lambda_1 = -\operatorname{Re} g_{2p}(-i)$$

$$= -\frac{1}{\rho g}\sqrt{\frac{\pi}{2\gamma}}\sin\left[\frac{w^2 a}{g} - \frac{1}{\pi}\int_0^\infty \ln\left(\frac{v^{\pi/\gamma}-1}{v^{\pi/\gamma}}\,\frac{v^2}{v^2-1}\right)\frac{dv}{v^2-1}\right]$$

$$+\frac{1}{\pi\rho g}\int_0^\infty d\eta\left[\frac{\eta^{2\gamma/\pi}-1}{(\eta-1)(\eta^{2\gamma/\pi}-1)}\right]^{1/2}\cos\left[\frac{w^2 a}{g}\eta^{\gamma/\pi}\right.$$

$$\left.-\frac{1}{\pi}\int_0^\infty \ln\left(\frac{v^{\pi/\gamma}-1}{v^{\pi/\gamma}}\,\frac{v^2}{v^2-1}\right)\frac{\eta^{\gamma/\pi}dv}{v^2-\eta^{2\gamma/\pi}}\right]$$

$$\lambda_2 = \operatorname{Im} g_{2p}(-i)$$

$$= -\frac{1}{\rho g}\sqrt{\frac{\pi}{2\gamma}}\cos\left[\frac{w^2 a}{g} - \frac{1}{\pi}\int_0^\infty \ln\left(\frac{v^{\pi/\gamma}-1}{v^{\pi/\gamma}}\,\frac{v^2}{v^2-1}\right)\frac{dv}{v^2-1}\right]$$

$$-\frac{1}{\pi\rho g}\int d\eta\left[\frac{\eta^{2\ /\pi-1}}{(\eta-1)(\eta^{2\gamma/\pi}-1)}\right]^{1/2}\sin\left[\frac{w^2 a}{g}\eta^{\gamma/\pi}\right.$$

$$\left.-\frac{1}{\pi}\int_0 \ln\left(\frac{v^{\pi/\gamma}-1}{v^{\pi/\gamma}}\,\frac{v^2}{v^2-1}\right)\frac{\eta^{\gamma/\pi}dv}{v^2-\eta^{2\gamma/\pi}}\right] .$$

As $z \longrightarrow \infty$, we see that, from (5.8) and (5.9)

$$\Phi \longrightarrow -\operatorname{Re}\left\{iI(-i)\lambda_2\, e^{i(wt-\frac{w^2}{g}z)}\right\}$$

$$= -\lambda_2 \sqrt{\frac{2\gamma}{\pi}}\; e^{\frac{w^2}{g} y} \cos(wt - \frac{w^2}{g}x + \alpha) ,$$

where

$$\alpha = \frac{1}{\pi}\int_0^\infty \ln\left( \frac{v^{\pi/\gamma} - 1}{v^{\pi/\gamma}} \; \frac{v^2}{v^2-1} \right) \frac{dv}{v^2-1} .$$

We remark here that the steady-state solution $\Phi$ found here is only valid for $\operatorname{Re} z > a$; however, we can always find an analytical continuation of $\Phi$ for $\operatorname{Re} z < a$. In the next section, we shall consider the case of a $45^\circ$ beach and show how a solution for the whole sector can be constructed.

## 6. Steady-state problem for the case of a $45^\circ$ beach.

The steady-state solution for $\gamma = \frac{\pi}{4}$ can be obtained directly from the solution for general angles given in Sec. 5 However, we prefer to treat this special case separately in order to compare what we find with the solution derived from the transient problem in Sec. 4. The method developed here can also be applied to the cases of $\gamma = \frac{\pi}{2m}$, $m =$ positive integer.

We recall that the functional equations which $g_1(\zeta)$ and $g_2(\zeta)$ satisfy are

$$(6.1) \qquad (\zeta + i)\, g_1(\zeta) = (\zeta - i)\, g_1(\zeta e^{i2\gamma})$$

$$(6.2) \qquad g_2(\zeta) = g_2(\zeta e^{i2\gamma}) - \frac{2i\zeta\, e^{-\frac{w^2 a\zeta}{g}}}{\rho g(\zeta+i)g_1(\zeta)} ;$$

for $\gamma = \frac{\pi}{4}$,

$$g_1(\zeta) = \frac{\zeta^2}{(\zeta+1)(\zeta+i)} ,$$

as can be easily verified. We have, from (5.7),

$$\Phi = - \mathrm{Re}\left\{ \frac{\sin wt}{2\pi i}\int_P e^{\frac{w^2 z}{g}\zeta} \frac{\zeta}{(\zeta+1)(\zeta+i)} [g_{2p}(\zeta) + \lambda_1] d\zeta \right.$$

$$\left. + \frac{\cos wt}{2\pi i}\int_P e^{\frac{w^2 z}{g}\zeta} \frac{\lambda_2 \zeta}{(\zeta+1)(\zeta+i)} d\zeta \right\} ,$$

where

$$(6.4) \qquad g_{2p}(\zeta) = - \frac{4}{\pi\rho g}\int_0^\infty \frac{\eta^2(\eta+1)\exp(-\frac{w^2 a}{g}\eta)d\eta}{\eta^4 - \zeta^4}$$

$$0 \leq \arg \zeta \leq 2\gamma .$$

We deform $P$ to a path as shown in Fig. 4, and

$$(6.5) \qquad \Phi = - \mathrm{Re}\left\{ \sin wt\left[ \frac{1}{2\pi i}\int_0^\infty e^{-\frac{w^2 z}{g}r} \frac{r}{(r-1)(r-i)}\left[g_{2p}(re^{i\pi}) - \right.\right.\right.$$

$$g_{2p}(re^{-i\pi})\right] dr$$

$$+ e^{-\frac{iw^2z}{g}} \frac{(1-i)}{2} [ g_{2p}(-i) + \lambda_1] + e^{-\frac{w^2z}{g}} \frac{(1+i)}{4}[g_{2p}(e^{i\pi}) + g_{2p}(e^{-i\pi})$$

$$+ 2\lambda_1]\} + \cos wt[e^{-\frac{iw^2z}{g}} \frac{(1-i)}{2} \lambda_2 + e^{-\frac{w^2z}{g}} \frac{(1+i)}{2} \lambda_2]\} .$$

From (6.2), (6.4), we find that

$$g_{2p}(re^{i\pi}) - g_{2p}(re^{-i\pi}) = \frac{2i}{\rho g r} [e^{-\frac{w^2a}{g}r}(r+1) + e^{-\frac{iw^2a}{g}r}(r-i)$$

$$+ e^{\frac{iw^2a}{g}r}(r+i) + e^{\frac{w^2a}{g}r}(r-1)] .$$

$$(6.6) \quad g_{2p}(e^{i\pi}) + g_{2p}(e^{-i\pi}) = 2g_{2p}(1) + \frac{2i}{\rho g}(2e^{-\frac{w^2a}{g}} + (1-i)e^{-\frac{iw^2a}{g}}$$

$$- (1+i) e^{\frac{iw^2a}{g}}) .$$

$$g_{2p}(-i) = g_{2p}(1) + \frac{2(1-i)}{\rho g} e^{\frac{iw^2a}{g}}$$

$$= - \frac{2i}{\rho g} e^{-\frac{w^2a}{g}} - \frac{4}{\pi\rho g}\int_0^\infty \frac{\eta^2(\eta+1) \exp(-\frac{w^2a}{g}\eta)d\eta}{\eta^4 - 1} +$$

$$+ \frac{2(1-i)}{\rho g} e^{\frac{iw^2a}{g}} .$$

Therefore,

$$\lambda_1 = -\mathrm{Re}\, g_{2p}(-i)$$

$$= \frac{4}{\pi\rho g}\int_0^\infty \frac{\eta^2(\eta+1)\exp(-\frac{w^2a}{g}\eta)d\eta}{\eta^4 - 1}$$

$$- \frac{2}{g}(\cos\frac{w^2a}{g} + \sin\frac{w^2a}{g}) .$$

$$\lambda_2 = \mathrm{Im}\, g_{2p}(-i)$$

$$= \frac{2}{g}(-e^{-\frac{w^2a}{g}} - \cos\frac{w^2a}{g} + \sin\frac{w^2a}{g}) .$$

As $z \longrightarrow \infty$, we easily see that

$$\Phi \longrightarrow \mathrm{Re}[\frac{(1-i)}{\rho g}(e^{-\frac{w^2a}{g}} + \cos\frac{w^2a}{g} - \sin\frac{w^2a}{g})\, e^{i(wt-\frac{w^2}{g}z)}] ,$$

which is exactly the same as (4.5). It is also found from (6.5) and (6.6) that

$$\Phi = -\mathrm{Re}\Big\{\frac{\sin wt}{\pi g}\int_0^\infty [\frac{1}{r-1}[(1+i)\exp(-\frac{w^2}{g}(z+a)r) +$$

$$\exp(-\frac{w^2}{g}(z+ai)r) + i\exp(-\frac{w^2}{g}(z-ai)r)] + \frac{1}{r-i}[-i\exp$$

$$(-\frac{w^2}{g}(z+0)r) + \exp(-\frac{w^2}{g}(z+a)r) + (1-i)\exp(\frac{w^2}{g}(z-ia)r)]dr$$

$$+ \frac{(1-i)}{2}\lambda_2 e^{i(wt-\frac{w^2}{g}z)} + e^{-\frac{w^2z}{g}}\frac{(1+i)}{2}[(g_{2p}(i) + \frac{i}{\rho g}(2e^{-\frac{w^2a}{g}}$$

$$+ (1-i)e^{-\frac{iw^2a}{g}} - (1+i)e^{\frac{iw^2a}{g}}) + \lambda_1)\sin wt + \lambda_2\cos wt]\Big\} .$$

Noting that

$$\lambda_1 + g_{2p}(1) = -\frac{2i}{\rho g} e^{-\frac{w^2 a}{g}} - \frac{2}{\rho g}\left(\cos\frac{w^2 a}{g} + \sin\frac{w^2 a}{g}\right),$$

and

$$\int_{-z}^{\infty \exp(i\theta)-z} \frac{e^{-\eta}}{\eta} d\eta = -\,\mathrm{Ei}(z) - i\pi, \qquad |\theta| \leq \pi/2 .$$

We finally obtain, for Re $z > a$,

$$\Phi = \mathrm{Re}\,\frac{\sin wt}{\pi\rho g}\Big\{(1+i)\exp\left(-\frac{w^2(z+a)}{g}\right)\mathrm{Ei}\left(\frac{w^2(z+a)}{g}\right)$$

$$+ \exp\left(-\frac{w^2(z+ia)}{g}\right)\mathrm{Ei}\left(\frac{w^2(z+ia)}{g}\right) + i\exp\left(-\frac{w^2(z-ia)}{g}\right)\mathrm{Ei}\left(\frac{w^2(z-ia)}{g}\right)$$

$$-i\exp\left(-\frac{iw^2(z+a)}{g}\right)\mathrm{Ei}\left(\frac{iw^2(z+a)}{g}\right) + \exp\left(-\frac{iw^2(z-a)}{g}\right)\mathrm{Ei}\left(\frac{iw^2(z-a)}{g}\right)$$

$$+ (1-i)\exp\left(-\frac{iw^2(z-ia)}{g}\right)\mathrm{Ei}\left(\frac{iw^2(z-ia)}{g}\right)\Big\}$$

$$+ \mathrm{Re}\,\frac{1}{\rho g} e^{iwt}\Big\{\Big(e^{-\frac{w^2 a}{g}} + \cos\frac{w^2 a}{g} - \sin\frac{w^2 a}{g}\Big)\Big[(1+i)e^{-\frac{w^2 z}{g}}$$

$$+ (1-i)e^{-\frac{iw^2 z}{g}}\Big]\Big\},$$

which is again exactly same as (4.4). Since $\mathrm{Ei}\left(\frac{iw^2(z-a)}{g}\right)$ is an analytic function of $z$ in the slit $z$-plane for $\pi > \arg(z-a) > -\pi$, the above expression can also be considered as an analytic continuation of $\Phi$ to the left of Re $z = a$ in the sector, and furnishes a solution for the whole sector $-\gamma < \theta < 0$, $0 < r < \infty$ .

## Appendix I

Let us assume that both $\tilde{\phi}$ and $\tilde{F}$ can be represented by Laplace integrals,

$$\text{(I.1)} \qquad \tilde{\phi} = -\mathrm{Re}\,\frac{1}{2\pi i}\int_P e^{z'\zeta}\,\frac{h(\zeta)}{\zeta}\,d\zeta\ .$$

$$\text{(I.2)} \qquad \tilde{F}(x') = \mathrm{Re}\,\frac{1}{2\pi i}\int_P e^{z'\zeta}\,F^*(\zeta)d\zeta, \qquad \theta = 0\ .$$

Since $e^{z'\zeta}$ is a solution of the two-dimensional Laplace equation, $\tilde{\phi}$ must also be one. We now need only to construct $h(\zeta)$ such that the two boundary conditions in (2.3) are satisfied. Substitution of (I.1), (I.2) in the boundary condition at $\theta = 0$ in (2.3) gives

$$\text{(I.3)} \quad -\mathrm{Re}\,\frac{1}{2\pi i}\int_P e^{z'\zeta}\,\frac{h(\zeta)}{\zeta}(1 + i\zeta)d\zeta = \mathrm{Re}\,\frac{1}{2\pi i}\int_P e^{z'\zeta}F^*(\zeta)d\zeta$$

$$\text{for} \quad \theta = 0,$$

$$\text{(I.4)} \quad \mathrm{Im}\,\frac{1}{2\pi i}\int_P e^{z'\zeta}\,h(\zeta)e^{-i\gamma}\,d\zeta = 0, \text{ for } \theta = -\gamma\ .$$

We may rewrite (I.3) as follows:

$$\text{(I.5)} \quad \mathrm{Im}\,\frac{1}{2\pi i}\int_P e^{z'\zeta}\left[\,\frac{h(\zeta)}{\zeta}(\zeta - i) - iF^*(\zeta)\right]d\zeta = 0, \qquad \theta = 0\ .$$

To satisfy the condition (I.5), it will suffice to require that $h(\zeta)$ satisfies

$$\text{(I.6)} \quad \mathrm{Im}\left[\frac{h(\zeta)}{\zeta}(\zeta - i) - iF^*(\zeta)\right] = 0, \quad \text{for} \quad \arg\zeta = 0 .$$

Since $P$ is symmetric to the real axis in the $\zeta$-plane, it is easily shown that if (I.6) holds, so does (I.5). By virtue of the principle of reflection, we have, from (I.6)

$$\text{(I.7)} \quad \frac{h(\zeta)}{\zeta}(\zeta - i) - iF^*(\zeta) = \frac{\overline{h}(\zeta)}{\zeta}(\zeta + i) + iF^*(\zeta) ,$$

where we assume that $\overline{F^*(\zeta)} = F^*(\overline{\zeta})$. In order to derive a condition for $h(\zeta)$ from (I.4), we introduce a new variable $\xi = \zeta e^{-i\gamma}$ . It is obtained from (I.4) that

$$\mathrm{Im}\,\frac{1}{2\pi i}\int_{P\exp(-i\gamma)} e^{r'\xi} h(\xi e^{i\gamma})\,d\xi = 0, \quad r' \text{ real.}$$

We assume that the integral remains the same if $P\exp(-i\gamma)$ is rotated back to $P$ . Hence (I.4) will be satisfied if

$$\text{(I.8)} \qquad \mathrm{Im}\, h(\zeta) = 0, \quad \text{for} \quad \arg\zeta = \gamma .$$

By the principle of reflection, we have

$$\overline{h}(\zeta) = h(\zeta e^{i2\gamma}) .$$

Substitution of the above relation for $\overline{h}(\zeta)$ in (I.7) yields

$$(\zeta - i)h(\zeta) = (\zeta + i)h(\zeta e^{i2\gamma}) + 2iF^{*}(\zeta) ,$$

which is the same functional relation as (2.11) subjecto to the condition (I.8).

## Appendix II

We consider the equation

$$\text{(II.1)} \qquad h_2(\zeta) - h_2(\zeta\, e^{i2\gamma}) = \frac{2i\zeta F^*(\zeta)}{(\zeta-i)h_1(\zeta)} \quad .$$

Set

$$\zeta = \eta^{\gamma/\pi}, \qquad h_2(\eta^{\gamma/\pi}) = H_2(\eta),$$

$$\frac{2i\eta^{\gamma/\pi}F^*(\eta^{\gamma/\pi})}{(\eta^{\gamma/\pi}-i)h_1(\eta^{\gamma/\pi})} = f(\eta),$$

then from (II.1), we have

$$H_2(\eta) - H_2(\eta e^{i2\pi}) = f(\eta) \ .$$

Suppose that $f(\eta)$ is absolutely integrable on the infinite interval $[0,\infty]$, and also satisfies Holder's Condition for real $\eta$. We define

$$\text{(II.2)} \qquad M_o(\eta) = \frac{1}{2\pi i}\int_0^\infty \frac{f(\xi)d\xi}{\zeta - \eta}$$

where $\eta$ is complex. Then for real $\eta$, the Plemelj formula [8] asserts that

$$M_o(\eta) = \frac{1}{2\pi i}\int_0^\infty \frac{f(\zeta)d\zeta}{\zeta - \eta} + \frac{1}{2} f(\eta) \ ,$$

$$M_o(\eta e^{i2\pi}) = \frac{1}{2\pi i}\int_0^\infty \frac{f(\xi)d\xi}{\xi - \eta} - \frac{1}{2} f(\eta) .$$

Hence for real $\eta$,

$$M_o(\eta) - M_o(\eta e^{i2\pi}) = f(\eta) .$$

Therefore, from (II.2),

$$\text{(II.3)} \qquad M_o(\zeta^{\pi/\gamma}) = \frac{1}{2\pi i}\int_0^\infty \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}}$$

furnishes a solution of (II.1) when $\zeta$ is real. However, (II.3) defines an analytic function for $0 \leq \arg \zeta \leq 2\gamma$, and can be analytically continued beyond the sector $0 \leq \arg \zeta \leq 2\gamma$ as follows: Let us consider paths $L_o$ and $L_1$ where $L_o$ is the real axis in the $\zeta$-plane and $L_1$ is the path obtained by rotating L. Counter-clockwise by an angle $\frac{\pi}{\gamma}\theta$, where $0 < \frac{\pi}{\gamma}\theta < 2\gamma$. We assume that no singularities of $f(\xi)$ is passed when $L_o$ is rotated to $L_1$, and $f(\zeta) \longrightarrow 0$ as $\zeta \longrightarrow \infty$ in the sector $0 \leq \arg \zeta \leq 2\gamma$. It follows from Cauchy integral formula that

$$\frac{1}{2\pi i}\int_{L_o} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}} = \frac{1}{2\pi i}\int_{L_1} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}} ,$$

where we restrict $\zeta$ in $\theta < \arg \zeta < 2\gamma$. As $\zeta \downarrow |\zeta|e^{i\theta}$, then

$$\frac{1}{2\pi i}\int_{L_0} \frac{f(\xi)}{\xi-|\zeta|^{\pi/\gamma}e^{i\pi\theta/\gamma}} = \frac{1}{2\pi i}\int_{L_1} \frac{f(\xi)d\xi}{\xi-|\zeta|^{\pi/\gamma}e^{i\pi\theta/\gamma}} +$$

$$\frac{1}{2} f(|\zeta|^{\pi/\gamma} e^{i\pi\theta/\gamma}) .$$

Since

$$M_1(\zeta) = \frac{1}{2\pi i}\int_{L_1} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}}$$

defines an analytic function for $\theta \leq \arg \zeta \leq \theta + 2\gamma$, and also is equal to $M_0(\zeta)$ when $\zeta = re^{i\theta}$, $M_1(\zeta)$ is an analytic continuation of $M_0(\zeta)$ in $\theta \leq \arg \zeta \leq \theta + 2\gamma$. For $\theta < 0$, we shift the path $L_0$ to $L_{-1}$ clockwise by an angle $\pi/\gamma\theta$, and make the same assumptions on $f(\zeta)$ as before. If we restrict $\zeta$ in the sector $0 \leq \arg \zeta \leq 2\gamma$, then

$$\frac{1}{2\pi i}\int_{L_0} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}} = \frac{1}{2\pi i}\int_{L_{-1}} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}} .$$

As $\zeta \downarrow |\zeta|$

$$\frac{1}{2\pi i}\int_{L_0} \frac{f(\xi)d\xi}{\xi-|\zeta|^{\pi/\gamma}} + f(|\zeta|^{\pi/\gamma}) = \frac{1}{2\pi i}\int_{L_{-1}} \frac{f(\xi)d\xi}{\xi-|\zeta|^{\pi/\gamma}} .$$

Since

$$M_{-1}(\zeta) = \frac{1}{2\pi i}\int_{L_{-1}} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}}$$

defines an analytic function in $\theta < \arg \zeta \leq \theta + 2\gamma$, and also equal to $M_o(\zeta)$ when $\zeta$ is real, $M_{-1}(\zeta)$ is an analytic continuation of $M_o(\zeta)$ below the real axis. The same process can be continued until Cauchy integral formula no longer holds. For further continuation, we can make use of the functional relation (II.1).

In the following, we shall show that the analytic function $M(\zeta) = \{ M_n(\zeta) \}$, $n = 0, \pm 1, \pm 2, \ldots$ does satisfy the functional relation (II.1). Without loss of generality, let us assume that $0 < \arg \zeta < 2\gamma$, then

$$M_o(\xi) = \frac{1}{2\pi i} \int_{L_o} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}} ,$$

$$M_1(\zeta e^{i2\gamma}) = \frac{1}{2\pi i} \int_{L_1} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma} e^{i2\pi}} = \frac{1}{2\pi i} \int_{L_1} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}} ,$$

where the angle between $L_o$ and $L_1$ is $\pi/\gamma\ \theta$ and $\theta < \arg \zeta + 2\gamma < \theta + 2\gamma$. Now we move $L_o$ to $L_1$. Since $\xi = \zeta^{\pi/\gamma}$ is a pole of the integrand in $M_o(\zeta)$, we have

$$M_o(\zeta) = \frac{1}{2\pi i} \int_{L_1} \frac{f(\xi)d\xi}{\xi - \zeta^{\pi/\gamma}} + f(\zeta^{\pi/\gamma}),$$

and it follows that

$$M_o(\zeta) - M_1(\zeta e^{i2\gamma}) = \frac{2i\zeta F^*(\zeta)}{(\zeta - i)h_1(\zeta)} .$$

Therefore $h_2(\zeta) = M(\zeta)$ is a solution of (II.1) . Next we show that $h_2(\zeta)$ satisfies the condition

$$\text{(II.4)} \qquad \mathrm{Im}\{ h_2(\zeta) \} = 0 \ , \quad \text{as} \quad \arg \zeta = \gamma \ .$$

Since $M_o(\zeta)$ is analytic in $0 \leq \arg \zeta \leq 2\gamma$ , it suffices to show that $M_o(\zeta)$ is real as $\arg \zeta = \gamma$ . The expression for $M_o(\zeta)$ is

$$M_o(\zeta) = \frac{1}{\pi} \int_{L_o} \frac{\xi^{\gamma/\pi} F^*(\xi^{\gamma/\pi}) d\xi}{(\xi^{\gamma/\pi} - i) h_1(\xi^{\gamma/\pi})(\zeta + |\zeta|)} \ ,$$

as $\arg \zeta = \gamma$. We recall that

$$h_1(\zeta) = \frac{\zeta + i}{\zeta} \exp\left\{ \frac{1}{\pi} \int_0^\infty \ell n \left( \frac{v^{\pi/\chi_1}}{v^{\pi/\gamma}} \frac{v^2}{v^2 - 1} \right) \frac{\zeta}{v^2 + \zeta^2} d\zeta \right\},$$

$$\text{for} \quad \pi/2 \leq \arg \zeta \leq \pi/2 \ ,$$

and $(\xi^{\gamma/\pi} - i) h_1(\xi^{\gamma/\pi})$ is real for real $\xi$ as seen from the above expression. Therefore, $M_o(\zeta)$ is real as $\arg \zeta = \gamma$ and this shows that the solution we have constructed for $h_2(\zeta)$ does satisfy the condition (II.4).

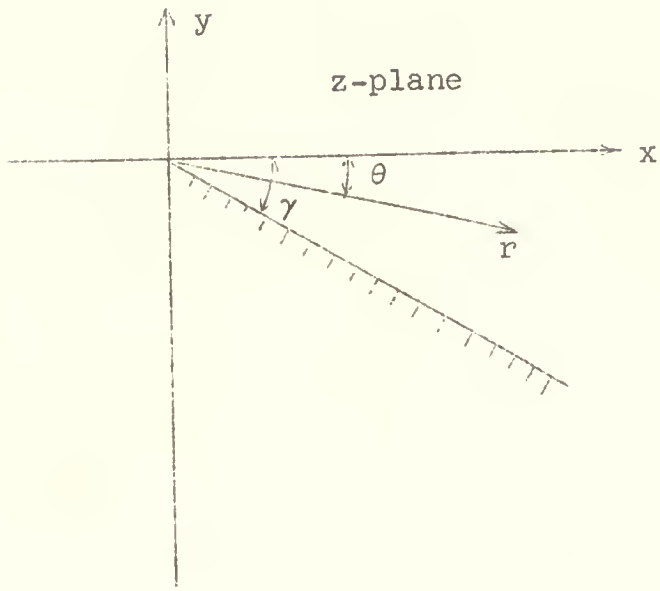
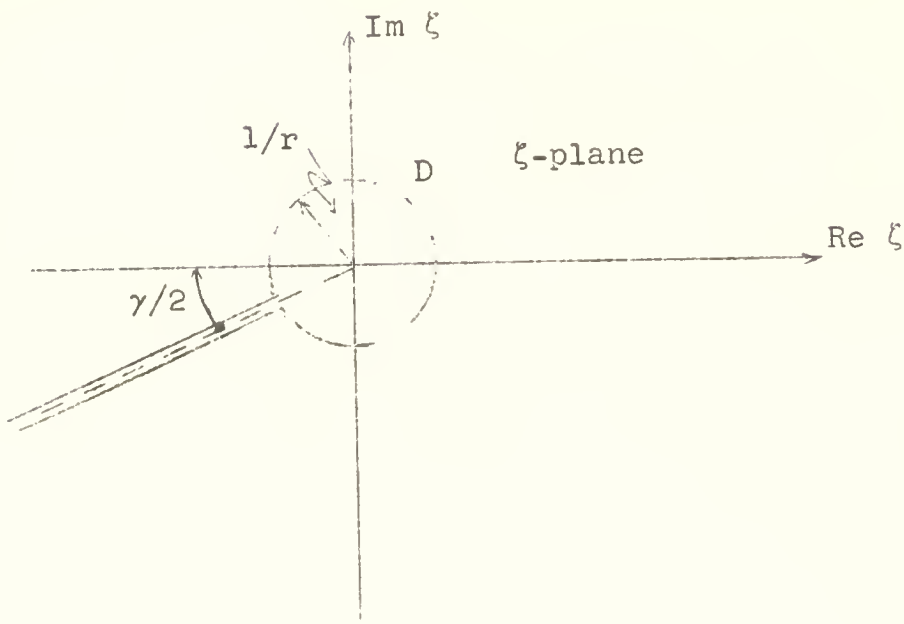



Fig. 1

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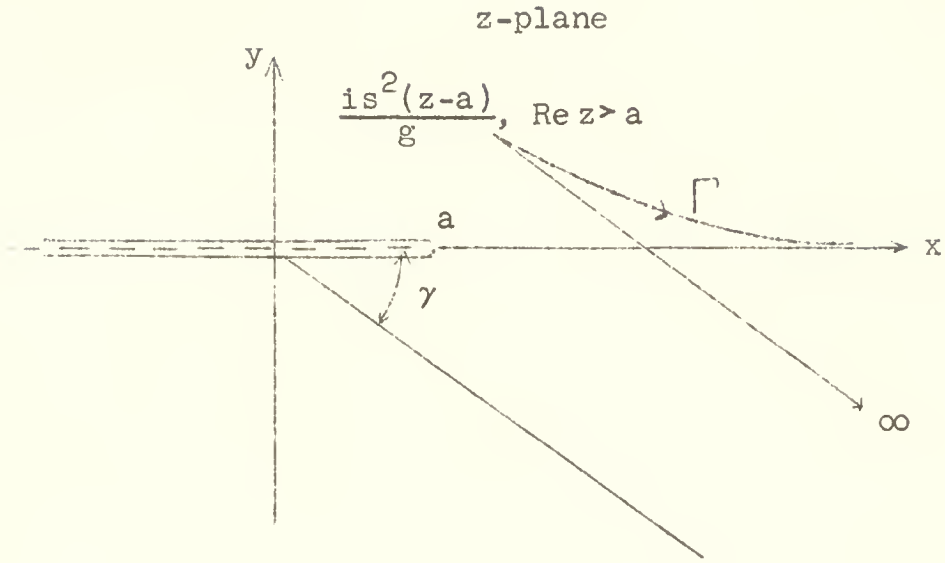


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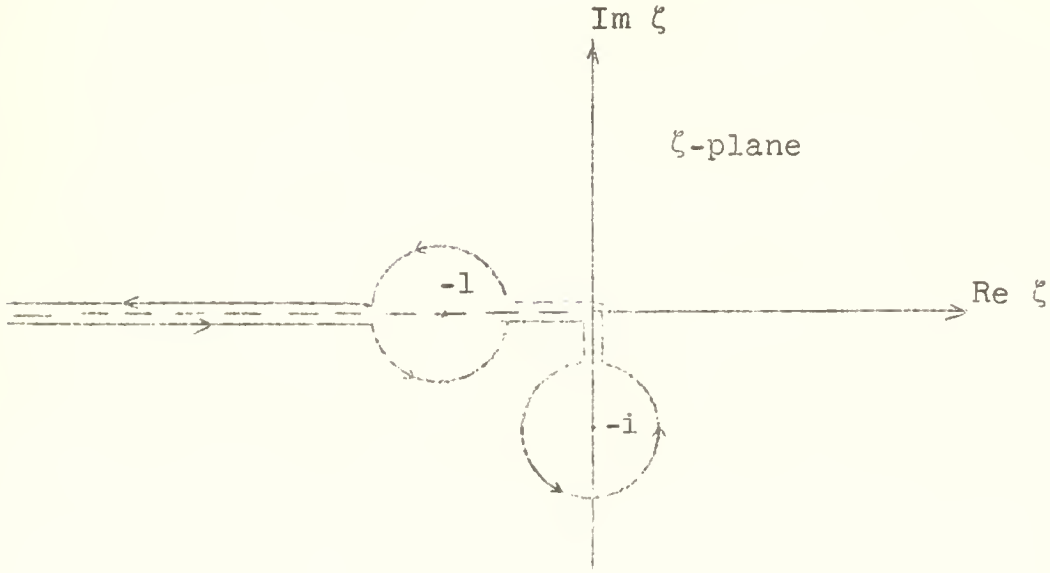


Fig. 4

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13 ABSTRACT

Two-dimensional water waves generated by a prescribed pressure disturbance on the equilibrium surface of a uniformly sloping beach is studied by means of the Laplace transform technique applied to the equations of the linear theory. Both transient and steady problems are considered, and in each case the problem reduces to solving an inhomogeneous q-difference equation. The arbitrary constants in the solutions of the transient and the steady-state problem are determined by respectively imposing a boundedness condition and a radiation condition at infinity. The example of an oscillatory point pressure acting on the equilibrium surface of a $45^{\circ}$ beach is worked out in detail. It seems easier to derive the steady-state solution from the transient problem than to find it directly.

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14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

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